Quantum dynamics of two capacitively coupled superconducting islands via Josephson junctions

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In this paper, we consider a system consisting of two capacitively coupled superconducting islands via Josephson junctions. We show that it can be reduced to two coupling harmonic oscillators under certain conditions, and solved exactly in terms of a displacing transformation, a beam-splitter-like transformation, and a squeezing transformation. It is found that the system evolves by a rotated-squeezed-coherent state when the system is initially in a coherent state. Quantum dynamics of the Cooper pairs in the two superconducting islands is investigated. It is shown that the number of the Cooper pairs in the two islands evolves periodically.

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Recent experiments [1, 2] have demonstrated the possibility of controlling or manipulating macroscopic quantum states in a single-Cooper-pair box, which consists of a small capacitance supuerconducting island coupled to a bulk superconductor via Josephson tunneling. Due to the small capacitance, there can be only zero or one excess Cooper pair on the island, which leads to an effective two-level system at appropriate values of external bias voltage. The possible use of these two-level systems as building blocks of quantum computer had been suggested already prior to the experimental work [3, 4]. However, it is clear that realizing such a quantum computer even with a modest number of qubits will prove exceedingly difficult and progress will be made only in small steps. A first step in this direction will consist in the coupling of two such Cooper-pair boxes.

Marquardt and Bruder [5] studied the quantum dynamics of a system which consists of a Cooper-pair box and a capacitively coupling large superconducting island, predicted generation of mesoscopically distinct quantum states. The large superconducting island can be described by a harmonic oscillator since it has a comparatively small charging energy, then we have E_J/E_C approache the infinity, where E_J and E_C denote the Josephson coupling energy and the charging energy of the large island, respectively. The coupling system of a Cooper-pair box and a large superconducting island is analogous to a system which consists of a two-level atom interacting with a single mode of the quantized electromagnetical field.

On the other hand, recently much attention has been paid to continuous-variable quantum information processing [6, 7, 8, 9, 10, 11, 12, 13] which is based on harmonic oscillators in infinite dimensional Hilbert spaces. Therefore, it is is of significance to study such systems of capacitively coupling large superconducting islands via Josephson junctions in order to explore continuous-variable quantum information processing realizations in solid-state systems. In this paper, we consider theoretically another possibility: two large superconducting islands are capacitively coupled via two Josephson junctions. Each island may be described by a harmonic oscillator, thus the system is reduced to a two-coupling-harmonic-oscillator system. We shall give an analytical solution of this system and investigate the quantum dynamics of the Cooper pairs in the islands.

Consider a system which consists of two capacitively coupled superconducting islands described in Fig. 1 where C_{J_i} and C_{g_i} are the capacitances of i-th Josephson junction and i-th gate capacitor, respectively, C_c is the coupling capacitance, and V_i denotes gate voltage. The total energy of this system is the sum of the charging energy and the Josephson coupling energy. The charging energy is given by

$$E_{ch}=\sum_{i=1}^{2}\left[\frac{Q_{J_i}^2}{2C_{J_i}}+\frac{Q_{g_i}^2}{2C_{g_i}}+V_i(Q_i-Q_{J_i})\right]+\frac{Q_c^2}{2C_c}, \quad (1)$$

where Q_i is the total charge on the i-th island, Q_c the charge on the coupling capacitor, and Q_{J_i} and Q_{g_i} denote the charges on i-th Josephson junction and i-th gate capacitor, respectively.

The charging energy can be expressed as a function of the total charges on the two superconducting islands Q_1 and Q_2 alone and the circuit parameters. If let $N_i = Q_i/2e$ denote the number of the Cooper pairs on the i-th island with e being the charge of the electron, then the charging energy can written as

$$\begin{aligned}E_{ch} &= \sum_{i=1}^{2}E_{C_i}(N_i-n_{g_i})^2+E_{12}(N_1-n_{g_1})(N_2-n_{g_2})\\ &\quad +N_i-\text{independent terms}, \end{aligned}\quad (2)$$

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where we have introduced the following notations

$$E_{12} = \frac{(2e)^2 C_C}{C_{t_1} C_{t_2} - C_C^2}, \tag{3}$$

$$E_{C_1} = \frac{(2e)^2 C_{t_2}}{2(C_{t_1} C_{t_2} - C_C^2)}, \tag{4}$$

$$E_{C_2} = \frac{(2e)^2 C_{t_1}}{2(C_{t_1} C_{t_2} - C_C^2)}, \tag{5}$$

$$C_{t_i} = C_{J_i} + C_{g_i} + C_C, \tag{6}$$

and the offsets introduced by the gates are given by

$$n_{g_i} = \frac{1}{2e}\left[C_{g_i} V_i - (-1)^i C_C (V_2 - V_1)\right]. \tag{7}$$

The Josephson coupling energy of the two superconducting islands is given by

$$E_{Jos} = -\sum_{i=1}^{2} E_{J_i} \cos\phi_i. \tag{8}$$

Quantum mechanically, the number of the Cooper pairs and the Josephson phase are regarded as interconjugate operators, and satisfy the boson communication relation $[\hat{\phi}_j, \hat{N}_k] = i\delta_{jk}$. Then we get the Hamiltonian

$$\hat{H} = \sum_{i=1}^{2}\left[E_{C_i}(\hat{N}_i - n_{g_i})^2 - E_{J_i}\cos\hat{\phi}_i\right] + E_{12}(\hat{N}_1 - n_{g_1})(\hat{N}_2 - n_{g_2}), \tag{9}$$

where we have discard the constant terms.

For a large superconducting island, the Josephson coupling energy is much larger than the charging energy, i.e., $E_{J_i}/E_{C_i} \to \infty$. In this case the approximation of a harmonic oscillator is valid, then one can replace $\cos\hat{\phi}_i$ term by the parabolic potential $1 - \hat{\phi}_i^2/2$. Hence the Hamiltonian (9) can be understood as that of two coupled harmonic oscillators. After making the following displacing transformation

$$\hat{d}_1(n_{g_1}, n_{g_2}) = \exp\left[i(\hat{\phi}_1 n_{g_1} + \hat{\phi}_2 n_{g_2})\right], \tag{10}$$

the Hamiltonian (9) can be expressed in terms of boson annihilation and creation operators as follows

$$\hat{H}_1 = \sum_{i=1}^{2}\hbar\omega_i \hat{a}_i^\dagger \hat{a}_i + \lambda\hbar(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) - \lambda\hbar(\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_1 \hat{a}_2), \tag{11}$$

where we have used the following decompositions

$$\hat{\phi}_i = \left(\frac{E_{C_i}}{2E_{J_i}}\right)^{1/4}\left(\hat{a}_i^\dagger + \hat{a}_i\right), \qquad (i = 1, 2), \tag{12}$$

$$\hat{N}_i = i\left(\frac{E_{J_i}}{8E_{C_i}}\right)^{1/4}\left(\hat{a}_i^\dagger - \hat{a}_i\right), \qquad (i = 1, 2), \tag{13}$$

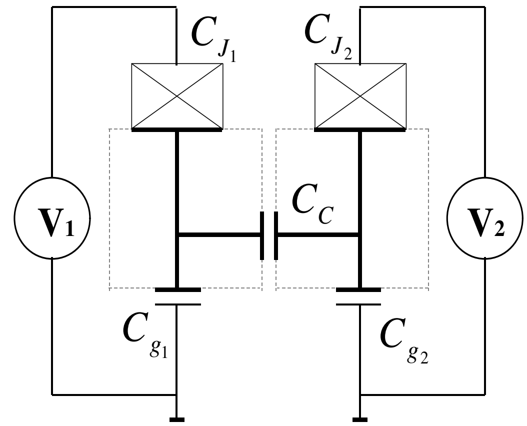

FIG. 1: Circuit diagram for the considered system. The two superconducting islands appearing as boxes corresponding to the regions marked by dashed rectangles. The junction and gate capacitances are C_{J_i} and C_{g_i} for i-island, respectively. The coupling capacitance is denoted by C_C and the gate voltages are V_1 and V_2.

and the free-evolution frequencies and the coupling constant are given, respectively, by

$$\omega_i = \frac{1}{\hbar}\sqrt{2E_{C_i}E_{J_i}}, \qquad (i = 1, 2), \tag{14}$$

$$\lambda = \frac{E_{12}}{2}\left(\frac{E_{J_1}E_{J_2}}{4E_{C_1}E_{C_2}}\right)^{1/4}. \tag{15}$$

Then the displacement operator (10) can be rewritten in terms of the annihilation and creation operators as

$$\hat{d}_1(n_{g_1}, n_{g_2}) = \prod_{i=1}^{2}\exp\left[\alpha_{0i}\hat{a}_i^\dagger - \alpha_{0i}^*\hat{a}_i\right] \equiv \hat{D}(\alpha_{01}, \alpha_{02}), \tag{16}$$

where α_{01} and α_{02} are given by

$$\alpha_{01} = i n_{g_1}\left(\frac{E_{C_1}}{2E_{J_1}}\right)^{1/4}, \alpha_{02} = i n_{g_2}\left(\frac{E_{C_2}}{2E_{J_2}}\right)^{1/4}. \tag{17}$$

For the sake of simplification, we consider the symmetric situation where $\omega_1 = \omega_2 \equiv \omega$ since $C_1 = C_2$ and $C_{J_1} = C_{J_2}$. In this case, the cross term in the Hamiltonian (11) can be gotten rid of by the following beam-splitter-like transformation over

$$\hat{B}(\varphi) = \exp\left[\varphi(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)\right]. \tag{18}$$

From (11) and (18) we obtain the transformed Hamiltonian

$$\begin{aligned}\hat{H}_2 &= \hat{B}^\dagger\left(\frac{\pi}{4}\right)\hat{H}_1\hat{B}\left(\frac{\pi}{4}\right) \\ &= \hbar\omega_1'\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_2'\hat{a}_2^\dagger\hat{a}_2 \\ &\quad +\frac{\hbar\lambda}{2}\left(\hat{a}_1^{\dagger 2}+\hat{a}_1^2\right) - \frac{\hbar\lambda}{2}\left(\hat{a}_2^{\dagger 2}+\hat{a}_2^2\right), \quad (19)\end{aligned}$$

where we have let $\omega_1' = \omega_1 - \lambda$ and $\omega_2' = \omega_2 + \lambda$. In the derivation of the Hamiltonian (19) we have used the following formula

$$\hat{B}^\dagger(\varphi)\hat{a}_1\hat{B}(\varphi) = \hat{a}_1\cos\varphi + \hat{a}_2\sin\varphi, \quad (20)$$

$$\hat{B}^\dagger(\varphi)\hat{a}_2\hat{B}(\varphi) = \hat{a}_2\cos\varphi - \hat{a}_1\sin\varphi. \quad (21)$$

Finally, we make a squeezing transformation over the Hamiltonian (19) to get that

$$\hat{H}_3 = \hat{S}^\dagger(\xi_1,\xi_2)\,\hat{H}_2\hat{S}(\xi_1,\xi_2) \quad (22)$$

where the squeezing transformation is defined by

$$\hat{S}(\xi_1,\xi_2) = \hat{S}_1(\xi_1)\hat{S}_2(\xi_2), \quad (23)$$

with $\hat{S}_i(\xi_i)$ being the single-mode squeezing operator defined by

$$\hat{S}_i(\xi_i) = \exp\left[-\frac{\xi_i}{2}\hat{a}_i^{\dagger 2} + \frac{\xi_i^*}{2}\hat{a}_i^2\right], \qquad (i=1,2). \quad (24)$$

It is easy to show that when the squeezing parameters are chosen as

$$\xi_i = \frac{1}{4}\ln\left(\frac{\omega_i' - (-1)^i\lambda}{\omega_i' + (-1)^i\lambda}\right), \qquad (i=1,2), \quad (25)$$

the squeezing transformation can diagonalize the Hamiltonian $\hat{H}_2$ as the following form

$$\hat{H}_3 = \hbar\Omega_1\hat{a}_1^\dagger\hat{a}_1 + \hbar\Omega_2\hat{a}_2^\dagger\hat{a}_2, \quad (26)$$

where the frequencies Ω_i is given by

$$\Omega_i = \sqrt{\omega_i'^2 - \lambda^2} = 2e\sqrt{\frac{E_J}{C_{t_i} - (-1)^i C_C}}, \qquad (i=1,2) \quad (27)$$

In the derivation of the Hamiltonian (23) we have used the following formula

$$\hat{S}_i^\dagger(\xi_i)\hat{a}_i\hat{S}_i(\xi_i) = \hat{a}_i\cosh\xi_i - \hat{a}_i^\dagger\sinh\xi_i, \quad (28)$$

$$\hat{S}_i^\dagger(\xi_i)\hat{a}_i^\dagger\hat{S}_i(\xi_i) = \hat{a}_i^\dagger\cosh\xi_i - \hat{a}_i\sinh\xi_i, \quad (29)$$

Assume that the two islands are initially in a coherent state

$$|\Psi(0)\rangle = \hat{D}(\alpha_1,\alpha_2)|0,0\rangle, \quad (30)$$

where $\hat{D}(\alpha_1,\alpha_2)$ is the displacement operator with respect to the two modes defined by Eq. (16).

After making three anti-transformations $\hat{S}^\dagger(\xi_1,\xi_2)$, $\hat{B}^\dagger(\pi/4)$, and $\hat{D}^\dagger(\alpha_{01},\alpha_{02})$, the initial state becomes

$$|\Psi(0)\rangle_3 = \hat{S}^\dagger(\xi_1,\xi_2)\hat{B}^\dagger\left(\frac{\pi}{4}\right)\hat{D}^\dagger(\alpha_{01},\alpha_{02})|\alpha_1,\alpha_2\rangle. \quad (31)$$

Then at an arbitrary time t, the wave function of the system is given by

$$|\Psi(t)\rangle_3 = \exp\left(-\frac{i}{\hbar}\hat{H}_3 t\right)|\Psi(0)\rangle_3. \quad (32)$$

Making use of the following formula

$$\hat{B}^\dagger\left(\frac{\pi}{4}\right)\hat{D}(\alpha_1-\alpha_{01},\alpha_2-\alpha_{02})\hat{B}\left(\frac{\pi}{4}\right) = \hat{D}(\gamma_1,\gamma_2), \quad (33)$$

where the two parameters γ_1 and γ_2 are defined as

$$\gamma_i = \frac{1}{\sqrt{2}}\left[(\alpha_1 + (-1)^i\alpha_2) - (\alpha_{01} + (-1)^i\alpha_{02})\right], \quad (34)$$

We can express the evaluation of the state as

$$|\Psi(t)\rangle_3 = \prod_{j=1}^{2}\left[\hat{R}_j\left(-i\Omega_j t\right)\hat{S}(-\xi_j)\hat{D}(\gamma_j)\right]|0,0\rangle. \quad (35)$$

which indicates that when the system is initially in a coherent state, the system evolves by a rotated-squeezed-coherent state in the transformed representation with a displacing transformation, a beam-splitter-like transformation, and a squeezing transformation. It should be mentioned that the state (35) is an entangled state in the original representation, although it is a non-entangled state in the transformed representation.

From (32) we can obtain the number of Cooper pairs on the i-th island

$$\begin{aligned}N_i &= n_{g_1} + \frac{1}{2}\left[\rho_2\sin(\Omega_2 t + \delta_2)\right. \\ &\quad \left. -(-1)^i\rho_1\sin(\Omega_1 t + \delta_1)\right], \qquad (i=1,2). \quad (36)\end{aligned}$$

with the amplitudes and phase shifts given by

$$\rho_i = \left(\frac{E_C}{32E_J}\right)^{\frac{1}{4}}\left[(\mathrm{Im}\gamma_i)^2 + e^{4\xi_i}(\mathrm{Re}\gamma_i)^2\right]^{1/2}, \quad (37)$$

$$\delta_i = \tan^{-1}\left(-e^{-2\xi_i}\tan\varphi_{\gamma_i}\right). \quad (38)$$

Form Eq. (36) it can be seen that the number of Cooper pairs on each island oscillates periodically, and one can control periods and amplitudes of these oscillations through changing circuit parameters and gate voltages.

The sum of Cooper pairs in the two islands is given by

$$\langle N_1 + N_2\rangle = (n_{g_1} + n_{g_2}) + \rho_2\sin(\Omega_2 t + \delta_2), \quad (39)$$

and the difference of Cooper pairs in the two islands is given by

$$\langle N_1 - N_2\rangle = (n_{g_1} - n_{g_2}) + \rho_1\sin(\Omega_1 t + \delta_1). \quad (40)$$

Making use of the wavefunction (35), we find that quantum fluctuation in each island is given by

$$\langle(\Delta\hat{N}_i)^2\rangle = \left(\frac{E_J}{32E_C}\right)^{\frac{1}{2}} \sum_{j=1}^{2}[\cos^2\Omega_j t + e^{4\xi_j}\sin^2\Omega_j t], \quad (41)$$

which implies that the quantum fluctuation of the Cooper pairs in each island oscillates periodically with the time evolution. The amplitude of the quantum fluctuation is dependent of the squeezing parameter ξ_i, i.e., the Josephson coupling energy and charging energy of the system under our consideration.

In summary, we have investigated quantum dynamics of two capacitively coupled superconducting islands via Josephson junctions. We have shown that the system under our consideration can be reduced to two coupling harmonic oscillators. We have obtained an exact solution of the system when two harmonic oscillators are initially in coherent states, and found that the system evolves by a rotated-squeezed-coherent state. We have also studied the Cooper-pair evolution and quantum fluctuations in the two superconducting islands and found that the number of the Cooper pairs in the two islands evolves periodically.

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